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DYNAMIC CHARACTERISTICS OF AN ELECTROMAGNETICALLY DRIVEN TRIGGER REGULATOR

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The dynamic characteristics of an electromagnetically driven trigger regulator with one pulse per period is considered for a model with one and a half degrees of freedom. The method of point transformations is used to find the decomposition of the parameter space into domains in each of which the system under investigation has the same qualitative structure of the decomposition of the phase space into trajectories. The parameter value ranges in which complex periodic motions occur in the system are established.

1. The dynamic model and equations of motion. The motion of electromechanical trigger regulators with electromagnetic drive [1 and 2] can be investigated with the aid of the following model. Oscillator *1* of soft-magnetic material (Fig. 1) oscillates under the action of the linear restoring force exerted by spring *2* and the pulses produced by interaction of the oscillator with pulse coil *3*. As the oscillator moves from left to right, pin *4* makes contact with leaf spring *5* near the position of static equilibrium, bends it, and closes the electrical circuit through contact *6*. During interaction of the magnetic field of pulse coil *3* with oscillator *1* (which is close to the coil at this instant), the latter received a mechanical impulse. As the oscillator continues to move forward (from left to right) leaf spring *5* slips out from under pin *4*, the circuit is broken, and the pulse ceases. The electrical circuit is not closed during the reverse motion of the oscillator.

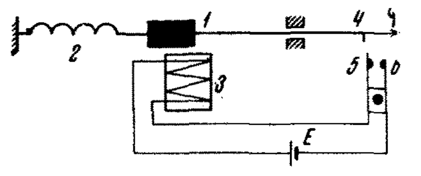

Fig. 1

We shall make the following simplifying assumptions.

1. There is no sparking when the electrical circuit is broken, and the circuit resistance, which is finite with the circuit closed, becomes infinite as soon as the circuit is opened [3].

2. Energy dissipation occurs through dry friction (the resistance of the leaf spring is negligible).

3. The impulse is proportional to the square of the current strength in the electrical circuit [1].

The equations of motion of the dynamic system are

$$m\varphi'' + k\varphi = -Q + M' y_1^2, \qquad L y_1' + R y_1 = E \tag{1.1}$$

when the electrical circuit is closed, and

$$m\varphi'' + k\varphi = -Q\frac{\varphi'}{|\varphi'|}, \qquad y_1 = 0 \tag{1.2}$$

when the circuit is open.

Here φ is the coordinate of the oscillator measured from the position of static equilibrium, m the mass of the oscillator, k the coefficient of elasticity of the oscillator spring, Q the coefficient of dry friction, My_1^2 the impulse, y_1 the current strength, L the self-inductance of the coil, R the circuit resistance, and E the electromotive force.

The electrical circuit closes for $\varphi = -\varphi_1, \varphi' \geqslant 0$ and opens for $\varphi = -\varphi_2,\ \varphi' > 0$ $(\varphi_1 > \varphi_2 > 0)$. Let us introduce the following variables and parameters t' is the initial time):

$$t = t'\left(\frac{k}{m}\right)^{1/2}, \qquad x = \frac{kR^2}{ME^2}\varphi, \qquad y = \frac{R}{E}y_1$$

$$a = \frac{R}{L}\left(\frac{m}{k}\right)^{1/2}, \qquad b = \frac{kR^2}{ME^2}\frac{\varphi_1 - \varphi_2}{2}, \qquad d = \frac{kR^2}{ME^2}\varphi_2, \qquad r = \frac{QR^2}{ME^2}$$

We can rewrite (1.1) and (1.2) as

$$x'' + x = -r + y^2,\ y' + ay = a \quad \text{for } x' \geqslant 0,\ |x + b + d| < b \tag{1.3}$$

$$x'' + x = -r\frac{x'}{|x'|}, \quad y = 0 \text{ for: } \begin{cases} x' \geqslant 0,\ |x + b + d| > b \\ \text{or} \quad x' < 0 \end{cases} \tag{1.4}$$

respectively.

Transition from (1.4) to (1.3) occurs when $x = -2b - d,\ x' > 0$, and transition from (1.3) to (1.4) when $x = -d,\ x > 0$.

The system is characterized by the four essential parameters a, b, d, r, which, by the physical meaning of the constituent parameters, can only be nonnegative. Let us set $x = z$ and assume that x, y, z are the Cartesian coordinates of the phase space of the dynamic system under consideration.

2. Reducing the problem to point transformations. The phase space of the dynamic system under discussion consists of a part of a plane and the attached three-dimensional domain. Moving in the plane $y = 0$, the representing point at some instant reaches the half-line Γ_1 $(x = -2b - d,\ y = 0,\ z > 0)$, with the coordinate $z = u$. It then passes from this point into the three-dimensional domain of the phase space between the planes $x = -2b - d$ and $x = -d$ and moves along the trajectories of system (1.3).

The trajectories of system (1.3) lie on the bent tubes whose intersections by planes $y =$const are circles of radius C_1. The equations of the tubes are

$$\left[x - 1 + r + \frac{2(1-y)}{1+a^2} - \frac{(1-y)^2}{1+4a^2}\right]^2 + \left[z - \frac{2a(1-y)}{1+a^2} + \frac{2a(1-y)^2}{1+4a^2}\right]^2 = C_1^2 \tag{2.1}$$

If the representing point moving along one of the trajectories of the upper half-space in one of surfaces (2.1) reaches the half-plane $x = -d,\ z > 0$ at some point $x = -d$, $y > 0,\ z = v$ (this is the only case we shall consider), then the motion of the representing point can be additionally defined in accordance with the adopted idealization : on arriving at the plane $x = -d$ the representing point instantly jumps along the line parallel to the y-axis to the half-line Γ_2 ($x = -d$, $y = 0$; $z > 0$.) The representing point then continues to move from the point with the coordinate $z = v$ in the plane $y = 0$ in accordance with Eqs. (1.4). If $z = v \geqslant v_0$, then the representing point arrives at the half-line Γ_1 ; if $z = v < v_0$, it arrives at the rest segment $-r < x < r$, $y = 0, z = 0$ along trajectories in the plane $y = 0$ (here v_0 is the coordinate of the point on the straight line Γ_2 through which the trajectory tangent to the half-line Γ_1 passes).

Investigation of the decomposition of the phase space into trajectories reduces to the analysis of the point transformation of the half-line Γ_2 into itself.

The transformation S_1 maps the point of the half-line Γ_1 with the coordinate $z = u$ along the trajectories of the upper half-space into a point with the coordinate $z = v$ in the plane $x = -d$. The transformation S_2 effects an instantaneous jump along the plane $x = -d$ onto Γ_2. For example, introducing the transit time τ as the parameter, we obtain the following expressions for the transformation S_1S_2 :

$$u = \frac{1}{\sin\tau}\,[2b\cos\tau + (r - d - 1)(1 - \cos\tau) + 2F(a, \tau) - F(2a, \tau)]$$

$$v = \frac{1}{\sin\tau}\,[2b - (r - d - 1)(1 - \cos\tau) - 2\Phi(a, \tau) + \Phi(2a, \tau)] \tag{2.2}$$

$$F(a, \tau) = \frac{1}{1 + a^2}(e^{-a\tau} - \cos\tau + a\sin\tau), \qquad \Phi(a, \tau) = \frac{1 - e^{-a\tau}(\cos\tau + a\sin\tau)}{1 + a^2}$$

The plane

$$r = 2b + d \quad \text{for} \quad b \leqslant b_0 \equiv 1 - \frac{1 + e^{-a\pi}}{1 + a^2} + \frac{1 + e^{-2a\pi}}{2(1 + 4a^2)}$$

and the surface

$$r = 1 + b + d - \frac{1 + e^{-a\pi}}{1 + a^2} + \frac{1 + e^{-2a\pi}}{2(1 + 4a^2)} \quad \text{for} \quad b \geqslant b_0 \tag{2.3}$$

(the two of which will henceforth be referred to as the surface $\{\alpha_1\}$) isolate the domain in question in the parameter space a, b, d, r for whose points the entire half-line Γ_1 is mapped onto the plane $x = -d$ along the trajectories of the upper half-space (the parameter τ in expression (2.2) varies in the range $0 < \tau \leqslant \tau_0 \leqslant \pi$)

The transformation S_3 maps the point of the half-line Γ_2 with the coordinate $z = v_1$ into the point of the half-line Γ_1 with the coordinate $z = u$ along the trajectory of system (1.4). The quantities v_1 and u are related by the equation

$$[v_1^2 + (r - d)^2]^{1/2} - [u^2 + (r - 2b - d)^2]^{1/2} = 4r \tag{2.4}$$

The transformation S_3 is effected in our domain for

$$v_1 \geqslant v_0 \equiv 2[(2r + b)(r + b + d)]^{1/2}$$

The transformation $T = S_1S_2S_3$ in the indicated domain of the parameter space maps the half-line Γ_2 into itself (maps v_1 into v). As in [4] we can show that there exist parameter values for which the trajectories effecting the transformation T lie on a Möbius strip.

Equations (2.2) and (2.4) can be written as

$$v_1 = \varphi(\tau), \qquad v = \psi(\tau) \tag{2.5}$$

In the neighborhood of the fixed point defined by the condition $\varphi(\tau^*) = \psi(\tau^*) \equiv v^\circ$ the transformation T and T^2 can be expressed in series form,

$$(T) \quad z' = w(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots \quad (\lambda \equiv a_1) \tag{2.6}$$

$$(T^2) \qquad z'' = w(z') = \lambda^2 z + b_2 z^2 + b_3 z^3 + \ldots \tag{2.7}$$

$$z = v_1 - v^\circ,\ z' = v - v^\circ, \qquad a_n = \frac{1}{n!}\left(\frac{d^n v}{dv_1^n}\right)_{\tau^*} \quad (n = 1, 2, \ldots)$$

Here b_n are certain polynomials of the coefficients of series (2.6).

In passing through the surface $\{\alpha_2\}$ (defined by the condition $\lambda + 1 = 0$) in the parameter space, the stability of the fixed point of the transformation T changes in accordance with the sign of the quantity $g_0 \neq 0$ (g_0 is the coefficient b_3 computed for $\lambda + 1 = 0$),

$$g_0 = -2(a_3 + a_2^2) \tag{2.8}$$

It either sprouts or is impinged on by the two stable or unstable fixed points of the transformation T^2 (e.g. see [5]).

3. Characteristics of the phase space. The parameter space. Let us consider the parameter space a, b, r ($d =$ const > 0). The surfaces $\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$ and $\{\alpha_4\}$ decompose the parameter space into domains in each of which the system described by Eqs. (1.3) and (1.4) has the same qualitative structure of decomposition of the phase space into trajectories.

The surface $\{\alpha_2\}$ is given by the equations

$$[v^2 + (r-d)^2]^{1/2} - [u^2 + (r - 2b - d)^2]^{1/2} = 4r$$

$$u - 2a[\Phi(a, \tau) - \Phi(2a, \tau)] + \frac{u\{4r + [u^2 + (r - 2b - d)^2]^{1/2}\}}{[u^2 + (r - 2b - d)^2]^{1/2}} = 0 \tag{3.1}$$

where u and v are given by (2.2).

The surface $\{\alpha_3\}$ (defined by the condition of passage of the two-turn limit cycle through the point $z = v_0$ of the half-line Γ_2) is given by the equation

$$[v^2(\tau_1) + (r-d)^2]^{1/2} - [u^2(\tau_2) + (r - d - 2b)^2]^{1/2} = 4r \tag{3.2}$$

Here

$$u(\tau_2) = \frac{1}{\sin\tau_2}[2b\cos\tau_2 + (r - d - 1)(1 - \cos\tau_2) + 2F(a, \tau_2) - F(2a, \tau_2)]$$

$$v(\tau_1) = \frac{1}{\sin\tau_1}[2b - (r - d - 1)(1 - \cos\tau_1) - 2\Phi(a, \tau_1) + \Phi(2a, \tau_1)]$$

and τ_1 and τ_2 are roots of the equations

$$2b\cos\tau_1 + (r - d - 1)(1 - \cos\tau_1) + 2F(a, \tau_1) - F(2a, \tau_1) = 0$$

$$2b - (r - d - 1)(1 - \cos\tau_2) - 2\Phi(a, \tau_2) + \Phi(2a, \tau_2) = v_0\sin\tau_2$$

respectively.

Computations show that the surface $\{\alpha_2\}$ contains a curve at whose points the quantity g_0 vanishes. The behavior of the system near this curve depends essentially on the sign of the quantity h_0 (h_0 is the coefficient b_5 in (2.7) computed for the conditions $\lambda + 1 = 0, g_0 = 0$) Depending on the signs of the quantities g_0 and h_0, the neighborhood of the simple fixed point of the transformation T can contain either one or two pairs of fixed points of the transformation T^2 (stable or unstable two-turn limit cycles in the phase space). There exists a bifurcation surface (we denote it by $\{\alpha_4\}$) on which the

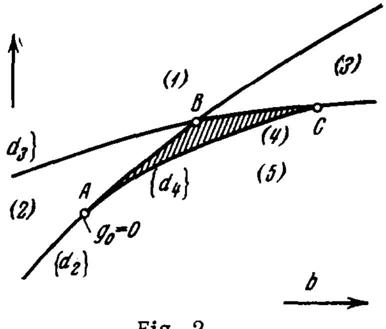

Fig. 2

two pairs of fixed points of the transformation T^2 merge and vanish [6]. The surface $\{\alpha_4\}$ emerges from the curve defined by the conditions $\lambda + 1 = 0$ and $g_0 = 0$.

We have not written out an analytic expression for the surface $\{\alpha_4\}$, but the latter can be approximated on the basis of the condition of merging of the two two-turn limit cycles. We constructed this surface on a BESM-3M computer for the parameter values $a = 2$, $d = 0.2$. Figure 2 is a qualitative diagram of the disposition of the intersections of the bifurcation surfaces $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_4\}$ bounding shaded domain (4). The following table gives the coordinates of the points of intersection of the indicated bifurcation surfaces:

	A	B	C
b	0.16445	0.16512	0.16572
r	0.058694	0.058860	0.059010

The widths Δr of this domain (4) for several values of b turned out to be

$b =$	0.16445	0.16465	0.16485	0.16505	0.16525	0.16545	0.16565
$\Delta r =$	0	$9.7 \cdot 10^{-8}$	$33.5 \cdot 10^{-8}$	$73.7 \cdot 10^{-8}$	$64.8 \cdot 10^{-8}$	$27.2 \cdot 10^{-8}$	$6.8 \cdot 10^{-8}$

Narrow domain (4) adjacent to the point B (the point of intersection of the bifurcation surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$) vanishes only if $b_i = 0$ ($i = 2, 3, \ldots$) in expression (2.7) for the parameter values corresponding to the point B. Depending on the sign of the quantity h_0, domain (4) lies [6] either "below" or "above" the point B. The parameter values $a = 2$, $d = 0.2$ are associated with the bifurcation corresponding to the case $h_0 < 0$.

Domain (4) is associated with a phase space with three limit cycles: a stable one-turn cycle, a stable two-turn cycle, and an unstable two-turn cycle, all of which lie on a Möbius strip. Figure 3 shows the projections of the cycles onto the plane xz. The twist in the Möbius strip occurs in the impulse zone

$$-2b - d \leqslant x < -d, \; 0 \leqslant y < 1, \; z > 0.$$

The broken curve represents the unstable two-turn limit cycle.

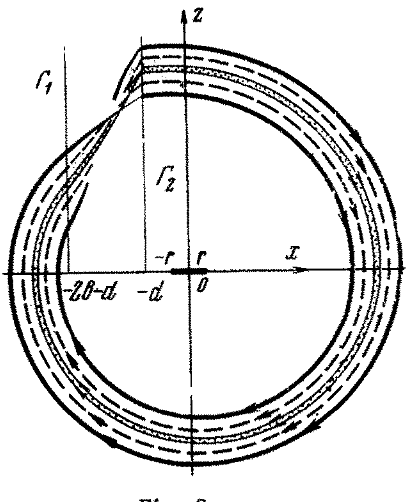

Fig. 3

The domain situated below the surface $\{\alpha_1\}$ and above the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ will be called domain (1); the domain lying above the surface $\{\alpha_2\}$ and below the surface $\{\alpha_3\}$ will be called domain (2); the domain below the surface $\{\alpha_2\}$ and above the surface $\{\alpha_3\}$ will be called domain (3); the domain below the surfaces $\{\alpha_2\}$, $\{\alpha_3\}$ and $\{\alpha_4\}$ will be called domain (5).

Domain (5) is associated with a phase space containing a stable one-turn limit cycle.

The stability of the one-turn limit cycle changes in the phase space on passage into domain (2) through the surface $\{\alpha_2\}$ in the parameter space; it sprouts the two-turn stable limit cycle in passing from domain (5) into domain (2) and is impinged on by the unstable two-turn limit cycle in passing from domain (4) into domain (2). For domain (2) the phase space contains a stable two-turn and an unstable one-turn limit cycle (Fig. 4).

The two-turn stable limit cycle vanishes in the phase space on passing from domain (4) to domain (3) through the surface $\{\alpha_3\}$ in the parameter space. In passing from domain

(5) to domain (3) in the phase space this boundary sprouts the two-turn unstable limit cycle. For domain (3) the phase space contains an unstable two-cycle and a stable one-turn limit cycle (Fig. 5).

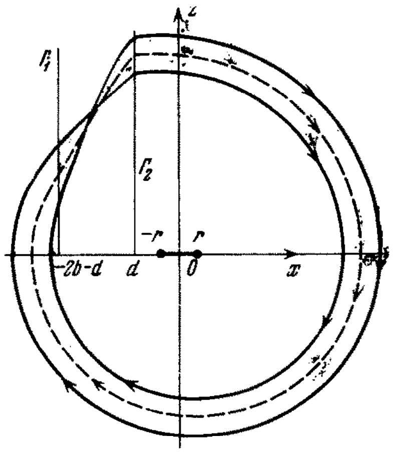

Fig. 4

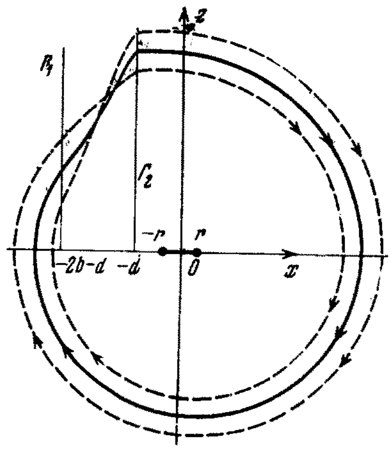

Fig. 5

The two-turn stable limit cycle vanishes (impinges on the boundary of the domain of attraction of the rest segment) on passing from domain (2) into domain (1) on the surface $\{\alpha_3\}$ in the parameter space. The unstable two-turn limit cycle merges with the one-turn stable limit cycle on passage from domain (3) into domain (1) on the surface $\{\alpha_2\}$. Near the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ in domain (1) the phase space contains an unstable limit cycle which vanishes with further increases in the parameter r. The unstable one-turn limit cycle lies on a Möbius strip and therefore does not decompose the phase space into parts on which the representing points move toward various attracting elements. Domain (1) is associated with a phase space in which the entire trajectory coils in towards the rest segment.

Here are some values of r corresponding to the intersections of the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ with the planes $a = 2$ and $d = 0.2$ calculated from Eqs. (3.1) and (3.2), respectively, for several values of b

$b = 0.05$	0.10	0.20	0.30	0.40	0.60	
$r = 0.0203$	0.0398	0.0669	0.0847	0.0972	0.1138	$\{\alpha_2\}$
$r = 0.0211$	0.0402	0.0668	0.0843	0.0967	0.1134	$\{\alpha_3\}$

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PERTURBATION OF NATURAL SMALL VIBRATION FREQUENCIES UPON INTRODUCTION OF DAMPING

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It is known that the complex natural frequencies $p_n = i\omega_n$ of a vibrating system take the form $p_n' = -\alpha_n + i\omega_n'$, $\alpha_n \geqslant 0$ upon the introduction of damping. It can be shown that under some condition the imaginary part of the complex frequency hance varies thus,

$$\omega_N' \leqslant \omega_N \text{ for } \omega_N > 0, \quad \omega_N' \geqslant \omega_N \text{ for } \omega_N < 0$$
$$|\omega_N| = \max_n |\omega_n| \tag{1}$$

The proof of the inequalities (1) follows from this lemma.

Lemma. Let $A > 0$, $B \geqslant 0$, $R \geqslant 0$ be self-adjoint $n \times n$ matrices, where the condition

$$(Rx, x)^2 < 4(Ax, x)(Bx, x)$$

(weak damping) is satisfied. Let $p_n = i\omega_n$ be the roots of the equation

$$\det(p^2 A + B) = 0 \tag{2}$$

and p_n' the roots of the equation

$$p^2 A + pR + B) = 0 \tag{3}$$

Let $|p_N| = \max_n |p_n|$. Then $p_N' = -\alpha_N' + i\omega_N'$, where $\alpha_N' \geqslant 0$, and the inequalities (1) are satisfied. Here p_N' denotes the root of (3) for which $|\omega_N'| = \max_n |\omega_N'|$.

Proof. Let x_N' be the eigenvector corresponding to the eigennumber p_n', that is

$$(p_N'^2 A + p_N' R + B) x_{N'} = 0 \tag{4}$$

Then

$$p_N'^2 (Ax_N', x_N') + p_N'(Rx_N', x_N') + (Bx_N', x_N') = 0$$

Hence

$$p_N' = \frac{-(Rx_N', x_N') \pm i\sqrt{4(Bx_N', x_N')(Ax_N', x_N') - (Rx_N', x_N')^2}}{2(Ax_N', x_N')} \tag{5}$$

Since $R \geqslant 0$, $A > 0$, then $\alpha_N' \geqslant 0$. Furthermore

$$\omega_N' = \left(\frac{(Bx_N', x_N')}{(Ax_N', x'_N)} - \frac{(Rx_N', x_N')^2}{4(Ax_N', x_N')^2}\right)^{1/2} \tag{6}$$

From the minimax principle it follows that

$$\omega_N^2 = \sup_x \frac{(Bx, x)}{(Ax, x)} \geqslant \frac{(Bx_N', x_N')}{(Ax_N', x_N')} \quad (\omega_N^2 = \max_n \omega_n^2). \tag{7}$$